Angle Addition Formulas

Consider two angles a and b in the following unit circle diagram such that

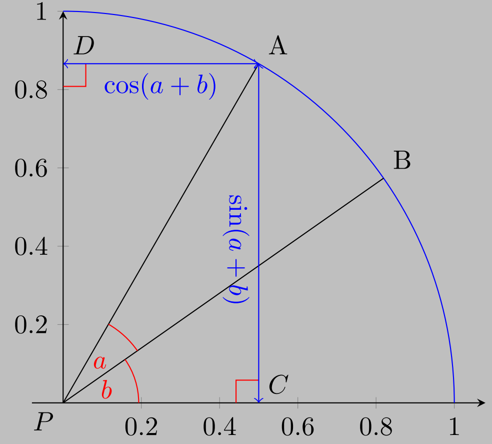
$$\angle APC \cong \angle(a+b)$$

and therefore

$$PC = DA = \cos(a+b)$$

and

$$PD = CA = \sin(a+b) \quad .$$

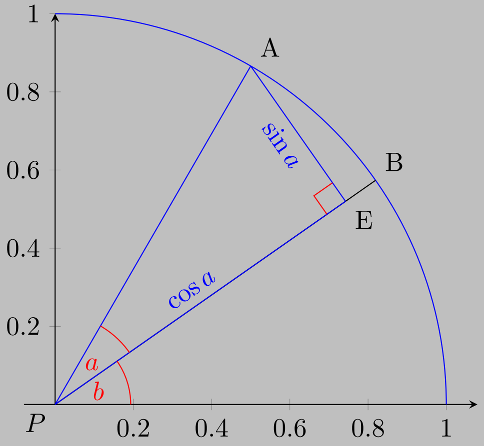


First we drop a perpendicular from A to $\overleftrightarrow{PB}$, creating point E. Since $\triangle PEA$ is a right triangle with a hypotenuse of length 1, we see that

$$PE = \cos a$$

and

$$EA = \sin a \quad .$$

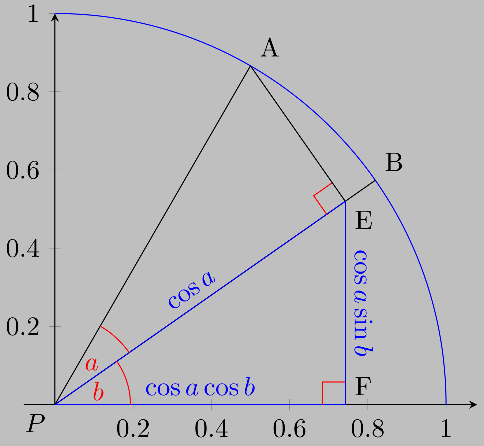


Next we drop a perpendicular from E to the x axis, creating point F. Since $\triangle PFE$ is a right triangle with a hypotenuse of length $\cos a$, we see that

$$PF = \cos a \cos b$$

and

$$FE = \cos a \sin b \quad .$$



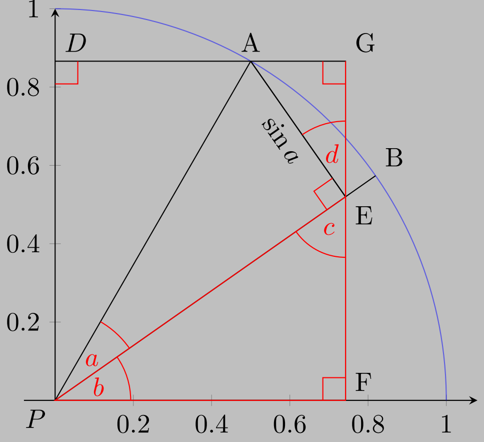
Now we intersect $\overleftrightarrow{FE}$ and $\overleftrightarrow{DA}$, creating point G and completing rectangle $PFGD$.

From $\triangle PFE$ we see that $\angle b$, $\angle c$ and $\angle PFE$ add to 180°.

Since $\overleftrightarrow{FG}$ is a straight line, $\angle d$, $\angle c$ and $\angle PEA$ add to 180°.

Since $\angle PFE$ and $\angle PEA$ are both right angles and therefore congruent,

$$\angle d \cong \angle b \quad .$$

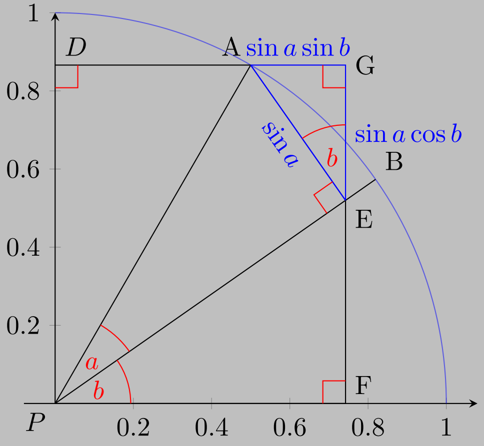


Since $\triangle EGA$ is a right triangle with a hypotenuse of length $\sin a$, we see that

$$EG = \sin a \cos b$$

and

$$AG = \sin a \sin b \quad .$$



Putting the pieces together, we have

$$AC = EG + EF$$

or

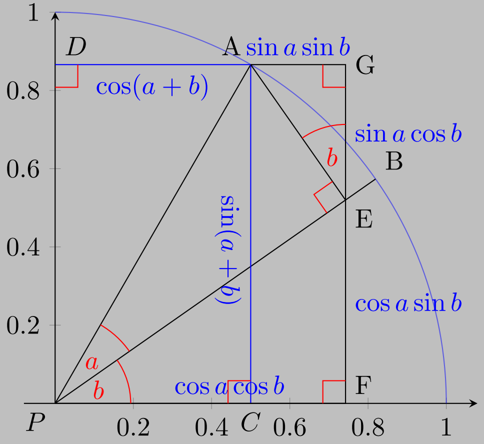
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

and

$$AD = PF - AG$$

or

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \quad .$$



Remember

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

Each formula contains one each of $\sin a$, $\cos a$, $\sin b$ and $\cos b$ in pairs of a with b.

$$\sin(a+0) = \sin(a)1 + \cos(a)0 = \sin a$$
$$\sin(0+b) = 0\cos b + 1\sin b = \sin b$$
$$\cos(a+0) = \cos(a)1 - \sin(a)0 = \cos a$$
$$\cos(0+b) = 1\cos b - 0\sin b = \cos b$$

Angle Subtraction Formulas

Since $\cos(-b) = \cos b$ and $\sin(-b) = -\sin b$ we can show

$$\begin{aligned} \sin(a-b) &= \sin a \cos(-b) + \cos a \sin(-b) \\ &= \sin a \cos b - \cos a \sin b \end{aligned}$$

and

$$\begin{aligned} \cos(a-b) &= \cos a \cos(-b) - \sin a \sin(-b) \\ &= \cos a \cos b + \sin a \sin b \quad . \end{aligned}$$

Since $15^\circ = 45^\circ - 30^\circ$ we can show

$$\begin{aligned} \sin 15^\circ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{1}{2}\right) \\ &= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right) \approx 0.259 \end{aligned}$$

and

$$\begin{aligned} \cos 15^\circ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \cos 30^\circ \\ &= \left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{1}{2}\right) \\ &= \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right) \approx 0.966 \quad . \end{aligned}$$

Also, $\sin 75^\circ = \cos 15^\circ$ and $\cos 75^\circ = \sin 15^\circ$.

Double Angle Formulas

$$\begin{aligned}\sin(2a) &= \sin(a+a)\\ &= \sin a\cos a + \cos a \sin a\\ &= 2\sin a\cos a\end{aligned}$$

and

$$\begin{aligned}\cos(2a) &= \sin(a+a)\\ &= \cos a\cos a - \sin a \sin a\\ &= (\cos a)^2 - (\sin a)^2\end{aligned}$$

Half Angle Formulas

Let $\theta = 2a$, so from the double angle formula we get

$$\cos\theta = (\cos\theta/2)^2 - (\sin\theta/2)^2 \quad .$$

Substituting $(\cos\theta/2)^2 = 1 - (\sin\theta/2)^2$ gives us

$$\cos\theta = 1 - 2(\sin\theta/2)^2$$
$$\cos\theta - 1 = -2(\sin\theta/2)^2$$
$$(1-\cos\theta)/2 = (\sin\theta/2)^2$$

and therefore

$$\sin\theta/2 = \pm\sqrt{(1-\cos\theta)/2}$$

where the sign depends on the quadrant.

Again from

$$\cos\theta = (\cos\theta/2)^2 - (\sin\theta/2)^2$$

substituting $(\sin\theta/2)^2 = 1 - (\cos\theta/2)^2$ gives us

$$\cos\theta = 2(\cos\theta/2)^2 - 1$$

$$1 + \cos\theta = 2(\cos\theta/2)^2$$

$$(1 + \cos\theta)/2 = (\cos\theta/2)^2$$

and therefore

$$\cos\theta/2 = \pm\sqrt{(1 + \cos\theta)/2}$$

where the sign depends on the quadrant.

We now have closed form solutions for trigonometric functions for $\theta = n(\pi/12) = n(15°)$ for any integer n.

Using the half angle formulas we can get closed form solutions for $\theta = n(\pi/24), n(\pi/48), n(\pi/96), \ldots$ for any integer n.

This is how Archimedes was able to approximate the circumference of a circle (2πr), starting with a hexagon and doubling the number of sides repeatedly.